On the Order of the Mertens Function

Tadej Kotnik and Jan van de Lune

CONTENTS

We describe a numerical experiment concerning the order of magnitude of $q(x) := M(x)/\sqrt{x}$, where $M(x)$ is the Mertens function (the summatory function of the Möbius function). It is known that, if the Riemann hypothesis is true and all non-trivial zeros of the Riemann zeta-function are simple, $q(x)$ can be approximated by a series of trigonometric functions of $\log x$. We try to obtain an Ω-estimate of the order of $q(x)$ by searching for increasingly large extrema of the sum of the first 10^2, 10^4, and 10^6 terms of this series. Based on the extrema found in the range $10^4 \leq x \leq 10^{10^{10}}$ we conjecture that $q(x) = \Omega_{\pm}(\sqrt{\log\log\log x})$.

1. INTRODUCTION

The Mertens function is defined as

$$M(x) = \sum_{1 \leq n \leq x} \mu(n)$$

where $\mu(n)$ is the Möbius function defined as $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is a product of k different primes, and $\mu(n) = 0$ if n contains a prime factor to a power higher than the first. The order of magnitude of the function $M(x)$ is closely related to the location of the zeros of the Riemann zeta-function, which is, largely due to its consequences for the distribution of the primes, one of the most important unsolved problems in analytic number theory. In short, a proof that $M(x) = O(x^\theta)$ would imply that $\zeta(s)$ has no zeros in the half-plane $\Re(s) > \theta$, and, consequently, that $\pi(x)$, the number of primes not exceeding x, can be approximated as

$$\pi(x) = \int_0^x \frac{du}{\log u} + O(x^\theta \log x).$$

This is the principal reason for the interest in the order of $M(x)$.

For convenience, we define

$$q(x) := M(x)/\sqrt{x}$$

and use $q(x)$ instead of $M(x)$ wherever it will simplify the formulations.

2000 AMS Subject Classification: Primary 11N56, 11Y35; Secondary 11-04

Keywords: Mertens function, Möbius function, Mertens hypothesis

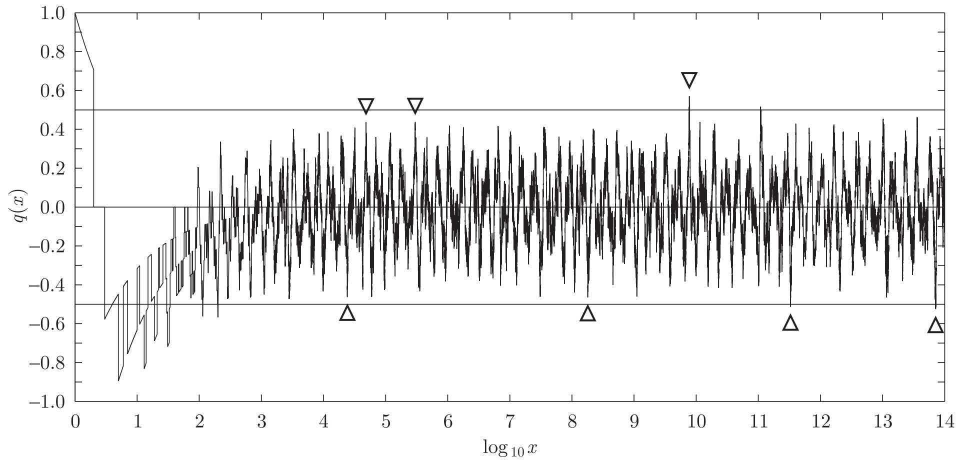

FIGURE 1. $q(x)$ in the range $1 \le x \le 10^{14}$. The triangles mark the ILq.

It is known that

$$q(x) = O(x^{1/2}), \qquad (1\text{–}1)$$

while the truth of the Riemann hypothesis would strengthen this to[1]

$$q(x) = O(x^{\varepsilon}) \quad \text{for every } \varepsilon > 0. \qquad (1\text{–}2)$$

It is also known that

$$q(x) = \Omega_{\pm}(1),$$

but it remains unknown whether $q(x)$ is unbounded, although many experts suppose this to be the case. Odlyzko and te Riele [Odlyzko and te Riele 85] have shown that $\limsup_{x\to\infty} q(x) > 1.06$ and $\liminf_{x\to\infty} q(x) < -1.009$, thereby disproving the famous conjecture by Mertens [Mertens 97] stating that $|q(x)| \le 1$. Pintz [Pintz 87] proved that $|q(x)| > 1$ occurs for some $x < e^{3.21\times 10^{64}} \simeq 10^{1.4\times 10^{64}}$.

Several authors have set forth conjectures stronger than the above O- and Ω-bounds of $q(x)$. We present and discuss these conjectures in Section 5.

Throughout this paper we assume that the Riemann hypothesis is true, and that all nontrivial zeros of the Riemann zeta-function are simple. Our aim will be to estimate computationally the order of $q(x)$ under these (and some additional) assumptions.

2. THE GENERAL BEHAVIOR OF $q(x)$

Figure 1 shows the plot of $q(x)$ in the range $1 \le x \le 10^{14}$, obtained by computing $M(x)$ for all integers in this range (see [Kotnik and van de Lune 03] for details). This plot suggests some guidelines for a computational estimation of the order of $q(x)$. First, $q(x)$ behaves quite atypically at small x; due to this we henceforth focus on $x \ge 10^4$. Second, the order of $q(x)$ is apparently rather small, with reliable estimates only likely to transpire at x much larger than 10^{14}. And third, it seems that there are certain bounds to the rate of variation of $q(x)$ on a logarithmic x-scale; more specifically, it seems that there exist constants G and $\beta_0 > 1$ such that for all $\beta \ge \beta_0$ and all sufficiently large x

$$\left| \frac{q(\beta x) - q(x)}{\log(\beta x) - \log x} \right| = \frac{|q(\beta x) - q(x)|}{\log \beta} \le G.$$

The values of $q(x)$ in the range $10^4 \le x \le 10^{14}$ suggest that for $\beta = 10^{0.0002}$ (the x-multiplier used in Section 4.3) and $x > 10^4$, we may quite safely take $G = 200$. Some additional numerical evidence for such boundedness of the rate of variation of $q(x)$ can be seen in Figure 2, and some theoretical ground for the dependence of q on $\log x$ is provided by the following result.

Theorem 2.1. (Titchmarsh.) *Let ζ denote the Riemann zeta-function, and $\rho = \frac{1}{2} + i\gamma$ its nontrivial zeros. Then there exists a sequence T_n, $n \le T_n \le n+1$, such that*

$$q(x) = 2 \lim_{n\to\infty} \sum_{0<\gamma<T_n} \operatorname{Re}\left(\frac{x^{i\gamma}}{\rho\,\zeta'(\rho)}\right) - R(x) + O(x^{-5/2})$$

where $R(x) = \frac{2-\mu(x)/2}{\sqrt{x}}$ if x is an integer, and $R(x) = \frac{2}{\sqrt{x}}$ otherwise.

This is a slight reformulation of Theorem 14.27 proved in [Titchmarsh 51]. The relation between q and $\log x$ is

[1]The actual strongest results to date are slight improvements of (1–1) and (1–2): [Ford 02] has shown that $q(x) = O(x^{1/2} \exp(-0.2098(\log x)^{3/5}/(\log\log x)^{1/5}))$, and [Titchmarsh 27] proved that the Riemann hypothesis implies $q(x) = O(x^{A/\log\log x})$ for some $A > 0$.

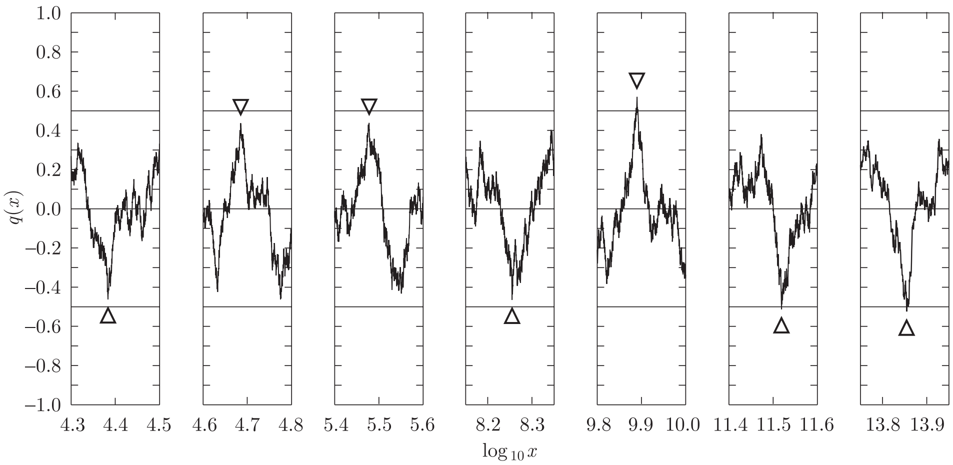

FIGURE 2. $q(x)$ in the neighborhood of the first seven ILq.

seen more clearly if the k-th term of the above series is rewritten as

$$|a_k| \cos(\gamma_k \log x + \arg a_k)$$

where $a_k = (\rho_k \zeta'(\rho_k))^{-1}$, with $\rho_k = \frac{1}{2} + i\gamma_k$ denoting the k-th ζ-zero in the upper half-plane, counted in the upward direction.

3. EXACT AND APPROXIMATE COMPUTATIONS OF $q(x)$

We now introduce a definition, which is admittedly somewhat *ad hoc*, but simple to state and to understand, and will adequately serve our purposes.

Definition 3.1. In the range $10^n < x \leq 10^{n+1}$, $n \in \mathbb{N}$, $n \geq 4$, we will refer to a q-value as an *increasingly large positive q* (abbreviated ILq$^+$) if it is the largest q-value in this range, and it exceeds all q in the range $10^4 \leq x \leq 10^n$. We introduce *increasingly large negative q* (ILq$^-$) analogously, and use the term *increasingly large q* (ILq) when referring to either of them.

There are seven ILq in the range $10^4 \leq x \leq 10^{14}$:

$$\begin{aligned} q(48433) &= +0.436215... \\ q(300551) &= +0.437776... \\ q(7766842813) &= +0.570591... \\ q(24185) &= -0.462977... \\ q(179919749) &= -0.464162... \\ q(330508686218) &= -0.512814... \\ q(71578936427177) &= -0.524797... \end{aligned}$$

The scarcity of the ILq suggests that a much broader x-range would have to be investigated for an estimation of the order of $q(x)$. Extending the systematic computations of $M(x)$ at $x \in \mathbb{N}$ would therefore very likely be futile for this purpose. However, the properties of the peaks of $q(x)$ containing the first seven ILq suggest that there may be a much more efficient way of detecting the ILq. As Figure 2 shows, these peaks are rather broad-based, and their widths on a logarithmic x-scale do not differ considerably, the latter observation being consistent with the considerations of Section 2. This suggests that large q-values may be detected efficiently by sampling $q(x)$ at sufficiently dense, yet *exponentially increasing* values of x. Together with a method for computing isolated values of $q(x)$, this would allow for a significant extension of the x-range in which to search for the ILq.

There are at least two algorithms for exact computation of isolated values of $q(x)$, one developed by Dress [Dress 93], and another by Deléglise and Rivat [Deléglise and Rivat 96]. A third such algorithm has been outlined in [Lagarias and Odlyzko 87]. But as x increases, these algorithms rapidly become too time- and memory-consuming. Thus, at the time of writing, a q-value at $x \sim 10^{20}$ is computable, but $q(10^{30})$ is definitely out of reach with any of these algorithms.

The remaining possibility is to compute the values of $q(x)$ approximately. Theorem 2.1 suggests that for any x, as K increases, partial sums of the form

$$q_K(x) := 2 \sum_{k=1}^{K} |a_k| \cos(\gamma_k \log x + \arg a_k) - R(x)$$

eventually converge to the value of $q(x)$.[2]

[2]Since $R(x) = O(x^{-1/2})$, in practice the computation of this correction term can be omitted for sufficiently large x.

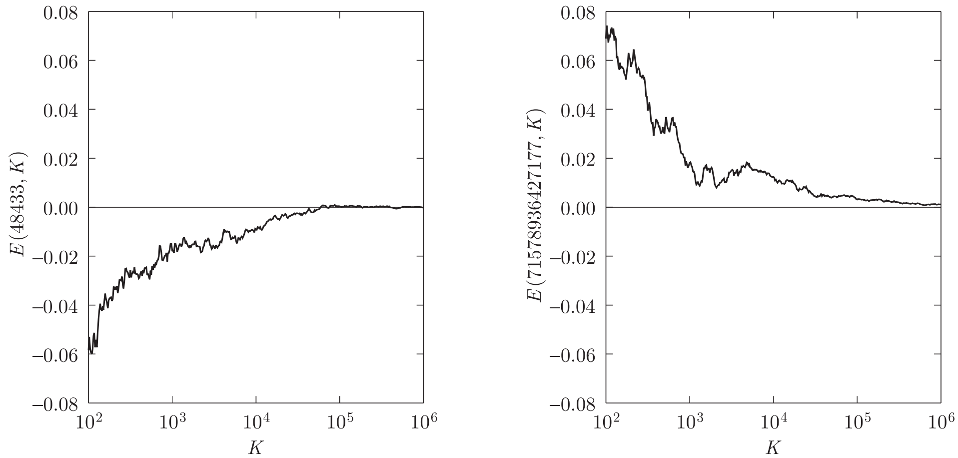

FIGURE 3. $E(x, K)$ as a function of K for the second and the seventh ILq.

In the range $10^4 \leq x \leq 10^6$, this convergence seems relatively rapid. Denoting

$$E(x, K) := q_K(x) - q(x),$$

we have $\left|E(x, 10^6)\right| < 0.0017$ for all integers in this range, as well as for each of the first seven ILq (see Table 1). Figure 3 shows $E(x, K)$ as a function of K for two of these ILq.

Since no reliable bounds can be imposed on $E(x, K)$ outside the x-range where the actual values of $q(x)$ are known, the use of $q_K(x)$ in estimating the order of $q(x)$ is in general open to doubt. However, it seems hard to envisage a mechanism that would consistently yield $|q_K(x)| > |q(x)|$ at x-values where $|q_K(x)|$ is relatively large, and numerical data also display nothing of the kind. In other words, it seems unlikely that *all* of the large $q_K(x)$ are overestimates of their respective $q(x)$. This suggests that for a fixed and sufficiently large K, a sufficiently comprehensive set of increasingly large $|q_K(x)|$ covering a sufficiently broad x-range could serve as a basis for an Ω-estimate of the order of $q(x)$. Using the same procedure to obtain also an O-estimate would be more questionable, since *some* of the large $q_K(x)$ are certainly underestimates of their respective $q(x)$, and this effect may become more pronounced as x increases.

4. THE SEARCH FOR INCREASINGLY LARGE q_K

4.1 Determination of γ_k, $|a_k|$, and $\arg a_k$

The list of the first million γ_k, accurate to $\pm 10^{-9}$, was kindly provided to the authors by Andrew M. Odlyzko. Mathematica 4.1 was used to improve the accuracy of these γ_k, and subsequently to compute the corresponding $|a_k|$ and $\arg a_k$. The γ_k were computed with an accuracy of $\pm 10^{-20}$ by setting `$MinPrecision` to 26 and applying the `FindRoot` routine to the `RiemannSiegelZ` function, with `AccuracyGoal` set to 20. The accuracy of the obtained values was verified by checking that `RiemannSiegelZ` changes sign between $\gamma_k - 10^{-20}$ and $\gamma_k + 10^{-20}$.

Using these values of γ_k, the values of $|a_k|$ and $\arg a_k$ were computed with an accuracy of $\pm 10^{-12}$. This was first done directly, using the `Zeta'` function, and for verification also indirectly, using the relations

$$|a_k| = \left(|Z'(\gamma_k)| \sqrt{\tfrac{1}{4} + \gamma_k^2}\right)^{-1};$$

$$\arg a_k = -\arctan(2\gamma_k) + \theta(\gamma_k) - (-1)^k \tfrac{\pi}{2}$$

and the `RiemannSiegelZ'` and `RiemannSiegelTheta` functions.

The search program for increasingly large q_K was written in Delphi 6.0 using the Int64 type (64 bits) for integer variables, and the Extended type (80 bits = 19–20 significant digits) for real variables. Due to the latter, the worst accuracy of γ_k in our computations was $\pm 10^{-14}$, while the accuracy of $|a_k|$ and $\arg a_k$ remained at $\pm 10^{-12}$. With these accuracies, with $\log x \leq 10^{10}$, and $K \leq 10^6$, the error in the computed q_K cannot exceed $\pm 6 \times 10^{-4}$.

4.2 The Range $10^4 \leq x \leq 10^{10000}$

At the seven ILq in the range $10^4 \leq x \leq 10^{14}$, the corresponding q_{10^2}, q_{10^4}, and q_{10^6}-values are also large. Furthermore, these large values are all detected by sampling q_{10^2}, q_{10^4}, and q_{10^6}, respectively, at exponentially increasing values of x, provided that the x-multiplier is sufficiently small (see Step (3) of the algorithm below and the comment that follows). The search for increasingly large values of q_K was performed for $K = 10^2$, 10^4, and 10^6. The algorithm used to cover the range

x	$M(x)$	$q(x)$	$q_{10^6}(x)$	$E(x,10^6)$
48433	96	0.436215	0.435930	–0.000285
300551	240	0.437776	0.437922	0.000146
7766842813	50286	0.570591	0.568908	–0.001683

x	$M(x)$	$q(x)$	$q_{10^6}(x)$	$E(x,10^6)$
24185	–72	–0.462977	–0.462869	0.000108
179919749	–6226	–0.464162	–0.462722	0.001440
330508686218	–294816	–0.512814	–0.511488	0.001326
71578936427177	–4440015	–0.524797	–0.523670	0.001127

TABLE 1. Top: The ILq^+ in the range $10^4 \le x \le 10^{14}$ and the corresponding q_{10^6}-values; bottom: The ILq^- in the range $10^4 \le x \le 10^{14}$ and the corresponding q_{10^6}-values

$10^4 \le x \le 10^{10000}$ for each of these three values of K can be presented schematically as follows:

(0) $x := 10^4$; $q_{K\,\min} := -0.35$; $q_{K\,\max} := 0.35$;

(1) compute $q_K(x)$ using the precomputed $|a_k|$, $\arg a_k$, and γ_k;
if not $(q_K(x) > 0.9\,q_{K\,\max}$ or $q_K(x) < 0.9\,q_{K\,\min})$ then go to (3);

(2) compute $q_K(\xi)$ for $\log_{10}\xi = \log_{10}x + 10^{-6}m$, $m = -199\ (1)\ 199$;
for each m: if $q_K(\xi) > q_{K\,\max}$ then $q_{K\,\max} := q_K(\xi)$; store $q_K(\xi)$;
for each m: if $q_K(\xi) < q_{K\,\min}$ then $q_{K\,\min} := q_K(\xi)$; store $q_K(\xi)$;

(3) multiply x by $10^{0.0002}$ and go to (1).

In analogy to the ILq, the set of stored values was then reduced by keeping only the largest value among those belonging to a range $10^n < x \le 10^{n+1}$, $n \in \mathbb{N}$. The values that remained in the set are given in Table 1. All the ILq in the range $10^4 \le x \le 10^{14}$ would have been detected by this algorithm (i.e., Step (2) would be run for the range containing each ILq), through either q_{10^2}, q_{10^4}, or q_{10^6}-values, also with the threshold in Step (1) set to 0.93 and the x-multiplier in Step (3) set to $10^{0.001}$.

4.3 The Range $10^{10000} < x \le 10^{10^{10}}$

At each of the increasingly large positive and negative q_{10^2}, q_{10^4}, and q_{10^6} found in the range $10^4 \le x \le 10^{10000}$, also for much smaller K the q_K-values are consistently large. Therefore, a preliminary qualifying requirement was introduced in the search for increasingly large q_{10^2}, q_{10^4}, and q_{10^6} in the range $10^{10000} < x \le 10^{10^{10}}$. Before Step (1) in the algorithm of Section 4.2, three preliminary checks were performed: $|q_5(x)| > 0.28$, $|q_{50}(x)| > 0.41$, and $|q_{500}(x)| > 0.48$ (for q_{10^4} and q_{10^6}). To also account for the fact that the maximal slope of q_5 on the logarithmic x-scale satisfies

$$\left|\frac{dq_5(x)}{d(\log x)}\right| \le 2\sum_{k=1}^{5}\gamma_k\,|a_k| = 8.716...,$$

a variable x-multiplier given by

$$\max\left(\exp\left(\frac{0.28-|q_5(x)|}{8.72}\right), 10^{0.0002}\right)$$

was used.

The set of stored values was again reduced by keeping only the largest value per each decimal order of magnitude, and the values that remained are given in Tables 2–4. None of the increasingly large positive or negative q_{10^2}, q_{10^4}, and q_{10^6}-values found in the range $10^{20} \le x \le 10^{10000}$ would have been missed by this algorithm, not even with the thresholds set at $|q_5(x)| > 0.31$, $|q_{50}(x)| > 0.46$, and $|q_{500}(x)| > 0.55$.

5. RESULTS AND DISCUSSION

In Figure 4, the ILq and the increasingly large positive and negative q_{10^2}, q_{10^4}, and q_{10^6}-values found in this study are plotted against $\sqrt{\log\log\log x}$. While all the q_{10^2} are within the angle defined by $\pm\frac{1}{2}\sqrt{\log\log\log x}$, some of the q_{10^4}, and even more of the q_{10^6} lie outside this angle, as do two of the ILq. If this trend persists, it would suggest that $\limsup_{x\to\infty} q(x)/\sqrt{\log\log\log x} \ge \frac{1}{2}$ and $\liminf_{x\to\infty} q(x)/\sqrt{\log\log\log x} \le -\frac{1}{2}$. Due to the extremely slow growth of $\sqrt{\log\log\log x}$, the studied x-range would have to be extended substantially to provide more insight into the actual situation. Merely for illustration, we note that if $\pm\frac{1}{2}\sqrt{\log\log\log x}$ were the actual asymptotic bounds of $q(x)$, then the first $|q(x)| > 1$ should occur not too far from $x \simeq 10^{2.3\times10^{23}}$, which is well below Pintz's bound $x \lesssim 10^{1.4\times10^{64}}$ (see Section 1).

$\log_{10} x$	$q_{10^2}(x)$
4.686776	0.386
5.476423	0.395
9.888887	0.513
42.353094	0.529
134.990791	0.544
320.901231	0.566
706.032905	0.594
1213.587635	0.607
5331.990640	0.708
693184.856510	0.708
2465089.468153	0.722
10827447.486503	0.731
10928392.830701	0.737
25462014.668048	0.737
40582074.348645	0.751
339249048.095259	0.754
380389486.067519	0.763
854362139.096477	0.781
1139774532.165446	0.784
1868381532.048425	0.805

$\log_{10} x$	$q_{10^2}(x)$
4.776083	–0.408
11.520454	–0.434
13.857640	–0.475
19.063485	–0.508
60.964153	–0.557
370.823174	–0.598
637.035857	–0.601
726.604947	–0.610
1305.385522	–0.615
11048.373075	–0.635
44512.589605	–0.648
67873.765888	–0.666
88251.095416	–0.671
331949.586837	–0.673
373684.562234	–0.703
593091.346546	–0.719
6664605.804812	–0.722
8412967.792159	–0.738
60987411.017141	–0.740
72619160.377213	–0.748
176081217.423035	–0.781
506751742.037025	–0.786
4698299201.556588	–0.793

TABLE 2. The increasingly large positive (left) and negative (right) q_{10^2}-values found in the range $10^4 \leq x \leq 10^{10^{10}}$.

$\log_{10} x$	$q_{10^4}(x)$
4.685212	0.430
9.890297	0.560
42.355325	0.612
706.031910	0.615
850.263123	0.620
873.623597	0.620
1176.714799	0.649
1213.586433	0.650
5331.990852	0.788
37548270.157211	0.796
108377624.910830	0.809
637358954.926941	0.813
825839004.998209	0.818
1670955708.587131	0.831
4519939603.762719	0.837

$\log_{10} x$	$q_{10^4}(x)$
4.775075	–0.461
11.519165	–0.501
13.855140	–0.514
19.064798	–0.579
60.963554	–0.587
370.823025	–0.670
1305.386211	–0.680
44512.589337	–0.688
61950.126978	–0.688
65153.059707	–0.699
77416.965980	–0.709
88251.096827	–0.732
201481.397575	–0.734
331949.587583	–0.736
373684.559306	–0.746
593091.345988	–0.754
2194019.447030	–0.754
3074103.225431	–0.764
5936921.848969	–0.776
8412967.791205	–0.778
24899895.454533	–0.797
72619160.376665	–0.819
176081217.424535	–0.834
1744552303.015566	–0.843

TABLE 3. The increasingly large positive (left) and negative (right) q_{10^4}-values found in the range $10^4 \leq x \leq 10^{10^{10}}$.

$\log_{10} x$	$q_{10^6}(x)$
4.685148	0.437
5.478152	0.438
9.890246	0.570
42.355231	0.619
427.468249	0.625
706.031885	0.627
850.263097	0.628
1176.714738	0.655
1213.585970	0.659
5331.990860	0.793
37548270.157201	0.804
108377624.910797	0.815
637358954.926958	0.817
825839004.998232	0.823
1670955708.587354	0.837
4519939603.762122	0.843

$\log_{10} x$	$q_{10^6}(x)$
4.383562	–0.463
11.519183	–0.511
13.854786	–0.524
19.064827	–0.585
60.963582	–0.589
370.823033	–0.677
1305.386175	–0.689
44512.589427	–0.696
65153.059738	–0.708
77416.965976	–0.715
88251.096825	–0.741
373684.559297	–0.753
593091.345965	–0.759
1201163.699231	–0.760
3074103.225118	–0.769
5936921.848867	–0.785
8412967.791234	–0.787
24899895.454588	–0.808
72619160.376650	–0.825
176081217.424585	–0.845
1744552303.015502	–0.851

TABLE 4. The increasingly large positive (left) and negative (right) q_{10^6}-values found in the range $10^4 \leq x \leq 10^{10^{10}}$.

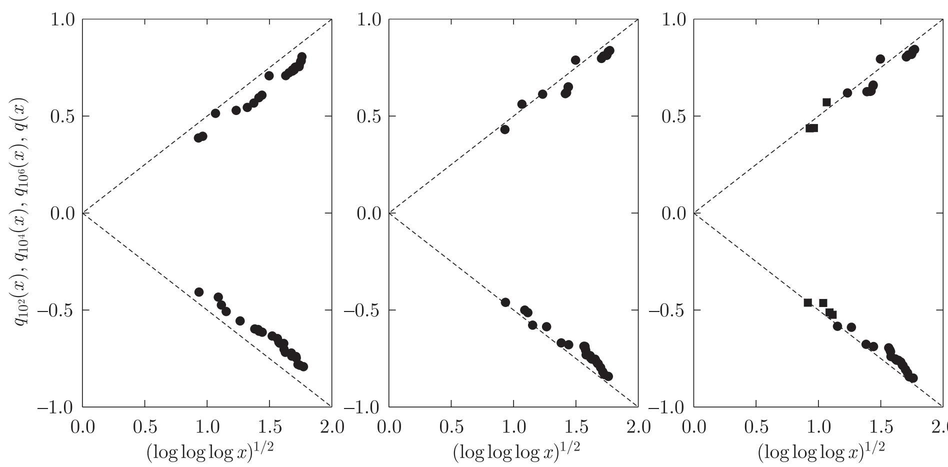

FIGURE 4. The increasingly large q_{10^2} (left), q_{10^4} (middle), q_{10^6} (right, circles), and the ILq (right, squares) plotted against $\sqrt{\log\log\log x}$. The dashed lines correspond to the functions $\pm\frac{1}{2}\sqrt{\log\log\log x}$.

In a slightly more conservative spirit, we have the following conjecture:

Conjecture 5.1. $q(x) = \Omega_{\pm}(\sqrt{\log\log\log x})$.

It is sensible to compare Conjecture 5.1 to some other conjectures about the order of $q(x)$ that have been set forth. We will omit here several conjectures that have already been disproved; some historical details about them can be found in a survey by [te Riele 93].

Good and Churchhouse [Good and Churchhouse 68], as well as Lévy in a comment to Saffari [Saffari 70], have proposed that

$$\limsup_{x\to\infty} \frac{|q(x)|}{\sqrt{\log\log x}} = C,$$

with $C = \frac{\sqrt{12}}{\pi}$ according to Good and Churchhouse, whereas $C = \frac{6\sqrt{2}}{\pi^2}$ according to Lévy. Either of these conjectures is much stronger than Conjecture 5.1, but unless the behavior of $q(x)$ changes drastically for very large x, it seems unclear how they could be reconciled with experimental data, since $\frac{\sqrt{12}}{\pi}\sqrt{\log\log x}$ (respectively $\frac{6\sqrt{2}}{\pi^2}\sqrt{\log\log x}$) exceeds the value of 1 at $x \simeq 10$ (respectively $x \simeq 48$), while $|q(x)|$ itself never exceeds even the value of 0.6 for any $x \le 10^{14}$. The conjectures of the form $q(x) = \Omega(\sqrt{\log\log x})$ are based on the assumption that the Möbius sequence $\{\mu(n)\}$ resembles a random sequence in its large-scale behavior, and their authors acknowledge that such probabilistic reasoning has as yet no firm theoretical foundation. Moreover, such assumptions of randomness seem questionable, due to the fact that on a sufficiently large scale, there is a clear regularity of the distribution of positive and negative values of $\mu(n)$. This large-scale regularity is quite apparent in the graph of $q(x)$ in Figure 1, where the presence of the first term of the series in Theorem 2.1, $|a_1|\cos(\gamma_1 \log x + \arg a_1) \approx 0.18\cos(14.14\log x - 1.69)$, is relatively easily discernible.[3]

More recently, [Ng 04], partly building on unpublished work by Gonek, conjectured that

$$\limsup_{x\to\infty}\frac{|q(x)|}{(\log\log\log x)^{5/4}} = B \qquad (5\text{–}1)$$

for some $B > 0$. This would be stronger than our Conjecture 5.1, as it replaces the power $\frac{1}{2}$ by $\frac{5}{4}$. In analogy to the right panel of Figure 4, Figure 5 shows the ILq and the increasingly large positive and negative q_{10^6}-values plotted against $(\log\log\log x)^{5/4}$. It seems hard to envisage how either the negative- or positive-valued points could outline an asymptote of the form $B(\log\log\log x)^{5/4}$, i.e., a straight line passing through the origin. This suggests either that as x increases, the q_{10^6}-values eventually become considerably smaller than the corresponding q-values, or that the power $\frac{5}{4}$ in (5–1) is an overestimate. The former is certainly possible, but it appears that also the latter cannot be excluded. Namely, the power $\frac{5}{4}$ stems from a further conjecture [Ng 04]

$$J(T) := \sum_{0<\gamma\le T}\frac{1}{|\rho\,\zeta'(\rho)|} \asymp (\log T)^{5/4}. \qquad (5\text{–}2)$$

At the one-millionth ζ-zero, where $T = 600269.677...$, the value of $J(T)/(\log T)^{5/4}$ is only 0.104..., and while this could be due to the small value of the multiplicative constant involved, it could also be due to the power $\frac{5}{4}$ in (5–2) being itself an overestimate.

[3]More rigorously, a few thousand samples of $q(10^w)$ covering the range $4 \le w \le 14$ suffice for the discrete Fourier transform to show a clear and very prominent peak at the frequency corresponding to γ_1. The peaks corresponding to the following γ_k are also clearly visible.

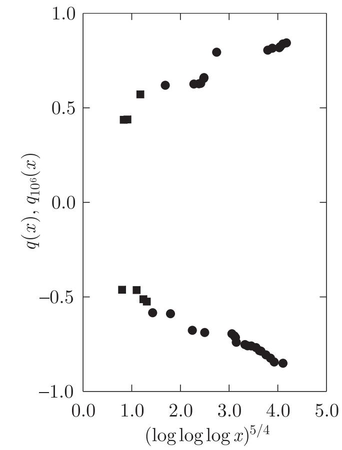

FIGURE 5. The ILq (squares) and increasingly large q_{10^6} (circles) plotted against $(\log\log\log x)^{5/4}$.

Finally, Conjecture 5.1 can also be considered with respect to the results obtainable for the similar yet somewhat simpler case involving the Chebyshev psi-function, for which [von Mangoldt 95] proved the formula

$$(x - \psi(x))/\sqrt{x} = 2\sum_{k=1}^{\infty}\left|\rho_k^{-1}\right|\cos\left(\gamma_k\log x + \arg\rho_k^{-1}\right) + O(x^{-1/2}).$$

Littlewood [Littlewood 17] has shown that

$$(x - \psi(x))/\sqrt{x} = \Omega_{\pm}(\log\log\log x), \qquad (5\text{–}3)$$

building his proof on the fact that the real part of $\sum_{k=1}^{\infty}\exp(i\rho_k z)/\rho_k$ is unbounded in the neighborhood of $z = 0$. An analogous proof of unboundedness cannot be provided for $q(x)$, since the real part of $\sum_{k=1}^{\infty}\exp(i\rho_k z)/(\rho_k\zeta'(\rho_k))$ is bounded in this neighborhood (see [Titchmarsh 51, Section 14.28]). In addition, although $\sum|\rho_k|^{-2}$ is known to be convergent, and the convergence of $\sum|\rho_k\zeta'(\rho_k)|^{-2}$ has not yet been proved, numerical data suggest that the latter series converges more rapidly, and to a smaller sum than the former (Table 5). And finally, Theorem 14.29B in [Titchmarsh 51] shows that there is also a plausible sufficient condition for the convergence of $\sum|\rho_k\zeta'(\rho_k)|^{-2}$. If all this is taken into account, it would perhaps not be too surprising if $q(x) = o((x - \psi(x))/\sqrt{x})$. We note in conclusion that this argument is not necessarily incompatible with (5–1), provided that (5–3) can be strengthened sufficiently.

ACKNOWLEDGMENTS

We would like to thank Andrew M. Odlyzko (University of Minnesota) for the list of the first two million zeros of the Riemann zeta-function, as well as Dr. ir. Herman J. J. te Riele

K	$\sum_{k=1}^{K} \lvert\rho_k\rvert^{-2}$	$\sum_{k=1}^{K} \lvert\rho_k \zeta'(\rho_k)\rvert^{-2}$
1	0.0049989...	0.0079462...
10	0.0135351...	0.0126558...
10^2	0.0199848...	0.0141436...
10^3	0.0223761...	0.0144539...
10^4	0.0229610...	0.0145073...
10^5	0.0230736...	0.0145155...
10^6	0.0230924...	0.0145167...
∞	$1 + \frac{\gamma}{2} - \log\sqrt{4\pi}$ $= 0.0230957...$	?

TABLE 5. Partial sums of $|\rho_k|^{-2}$ and $|\rho_k \zeta'(\rho_k)|^{-2}$; γ denotes Euler's constant.

(CWI, Amsterdam) and Prof. Roger Heath-Brown FRS (Oxford University) for their valuable comments and suggestions regarding the manuscript.

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Received May 4, 2004; accepted August 19, 2004.