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COMPUTATIONAL ESTIMATION OF THE CONSTANT $\beta(1)$ CHARACTERIZING THE ORDER OF $\zeta(1+it)$

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Abstract. The paper describes a computational estimation of the constant $\beta(1)$ characterizing the bounds of $|\zeta(1+it)|$. It is known that as $t \to \infty$

$$\frac{\zeta(2)}{2\beta(1)e^{\gamma}\,[1+o(1)]\log\log t} \leq |\zeta(1+it)| \leq 2\beta(1)e^{\gamma}\,[1+o(1)]\log\log t$$

with $\beta(1) \geq \frac{1}{2}$, while the truth of the Riemann hypothesis would also imply that $\beta(1) \leq 1$. In the range $1 < t \leq 10^{16}$, two sets of estimates of $\beta(1)$ are computed, one for increasingly small minima and another for increasingly large maxima of $|\zeta(1+it)|$. As t increases, the estimates in the first set rapidly fall below 1 and gradually reach values slightly below 0.70, while the estimates in the second set rapidly exceed $\frac{1}{2}$ and gradually reach values slightly above 0.64. The obtained numerical results are discussed and compared to the implications of recent theoretical work of Granville and Soundararajan.

1. Introduction

Denoting by $\zeta(\sigma+it)$ the Riemann zeta function, its restriction to the vertical line $\sigma = 1$ has a simple pole at $t = 0$ and no zeros. Away from this pole, $|\zeta(1+it)|$ is even and continuous, and both $|\zeta(1+it)|$ and $1/|\zeta(1+it)|$ are unbounded, so that as t increases, $|\zeta(1+it)|$ takes arbitrarily large values, as well as values arbitrarily close to zero. An illustration of this behavior is shown in Figure 1, and its more precise formulation is based on two inequalities due to Norman Levinson. Improving upon previous work by Bohr and Landau [1], Littlewood [2], [3], Titchmarsh [4], [5], and Chowla [6], Levinson showed in 1972 [7] that each of the two inequalities

$$(1) \qquad |\zeta(1+it)| \leq \frac{\zeta(2)}{e^{\gamma}(\log\log t - \log\log\log t)} \qquad \text{and} \qquad |\zeta(1+it)| \geq e^{\gamma}\log\log t$$

holds unconditionally for an infinite number of arbitrarily large values of t. In an arXiv preprint published in 2005, Granville and Soundararajan [8] report that in the denominator of the first of these inequalities the term $\log\log\log t$ can be improved to $O(1)$, while in the second inequality the term $\log\log t$ can be improved to $\log\log t + \log\log\log t - \log\log\log\log t + O(1)$. As will be discussed later, this second improvement is essential for the interpretation of the numerical data acquired in this paper.

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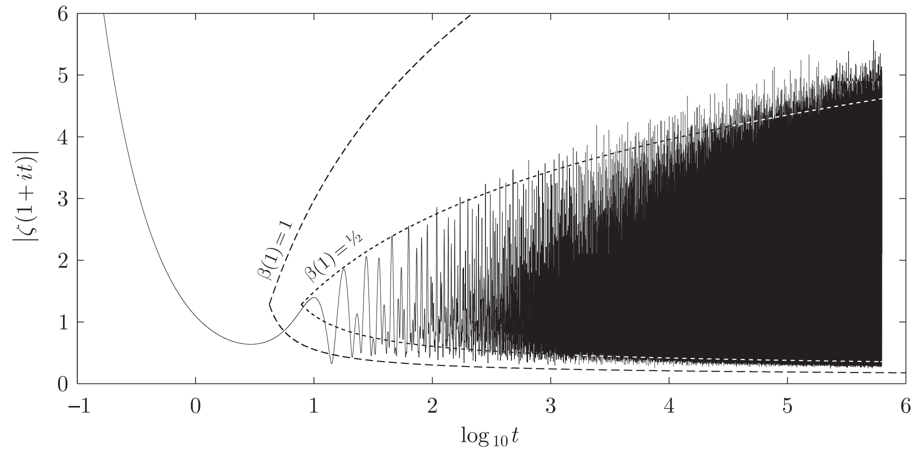

FIGURE 1

Littlewood proved in 1928 [3] that if the Riemann hypothesis is true, then for all sufficiently large t

$$\frac{\zeta(2)}{2\beta(1)e^{\gamma}\left[1+o(1)\right]\log\log t} \leq |\zeta(1+it)| \leq 2\beta(1)e^{\gamma}\left[1+o(1)\right]\log\log t \tag{2}$$

with $\frac{1}{2} \leq \beta(1) \leq 1$, where $\beta(1) := \lim_{\sigma\uparrow 1}\beta(\sigma)$, $\beta(\sigma) := \nu(\sigma)/(2-2\sigma)$, and $\nu(\sigma)$ denotes the lower bound of numbers a such that $\log\zeta(\sigma+it) = O(\log^a t)$ (further details on $\beta(\sigma)$ and $\nu(\sigma)$ can be found in Sections 14.3 and 14.9 of [10], but will not be of relevance here). The appearance of $\beta(1)$ on both the left- and the right-hand sides of (2) reflects a certain symmetry in the behavior of $|\zeta(1+it)|$ and $1/|\zeta(1+it)|$, showing that a strengthening of either the lower or the upper bound of $|\zeta(1+it)|$ implies a corresponding strengthening of the other bound.

Chowla showed in 1948 [6] that $\beta(1) \geq \frac{1}{2}$ holds unconditionally, which is also implied by (1). A comparison of (1) and (2) shows that under the Riemann hypothesis, only a factor of 2 remains undetermined in the asymptotic upper and lower bounds of $|\zeta(1+it)|$. In Figure 1 this is illustrated by the dashed curves, showing that at least in the plotted t-range, $|\zeta(1+it)|$ crosses both the lower and the upper bounds corresponding to $\beta(1) = \frac{1}{2}$. However, except in the neighborhood of its very first two minima, it remains well within both the upper and the lower bounds corresponding to $\beta(1) = 1$.

In principle, an extension of the plot of $|\zeta(1+it)|$ to sufficiently large t could be used for a numerical estimation of $\beta(1)$, but in practice this approach quickly reaches the limits of computational feasibility. As far as the estimation of $\beta(1)$ is concerned, such an approach would also generate a vast amount of unnecessary data, since only exceptionally small minima and exceptionally large maxima are of importance for this purpose. Still, a detailed insight into the early behavior of $|\zeta(1+it)|$ shown in Figure 1 provides two hints for a more efficient search for exceptional extrema. First, within the plotted t-range, there are 8 increasingly small minima (*ISm*) and 34 increasingly large maxima (*ILM*, excluding here and henceforth the singularity at $t=0$) of $|\zeta(1+it)|$. This suggests that also at larger t, the ISm and ILM should be abundant enough for a meaningful estimation of $\beta(1)$. Irrespective of the actual value of $\beta(1)$, in (2) the lower bound of $|\zeta(1+it)|$

is decreasing and the upper bound increasing, hence the restriction to the ISm and ILM seems a natural one. Second, Figure 1 suggests that it is sensible to keep the estimates at the ISm in one set, and those at the ILM in another one, since they behave quite differently: the estimate of $\beta(1)$ at the first ISm is larger than 1, and the subsequent ones gradually decrease, while the first few estimates at the ILM are smaller than $\frac{1}{2}$, and the subsequent ones gradually increase.

Section 2 describes two algorithms for a systematic search for ISm and ILM. The first one determines the smallest minimum and the largest maximum of $|\zeta(1+it)|$ in a given t-interval, and thus its application to sufficiently narrow adjacent intervals effectively yields the complete list of ISm and ILM in the t-range under consideration. The second algorithm is much faster and quite efficient in finding small minima and large maxima of $|\zeta(1+it)|$, but the lists obtained are not neccessarily complete (thus, the extrema found by this algorithm are referred to as *ISm candidates* and *ILM candidates*). Section 3 presents and discusses the estimates of $\beta(1)$ at the ISm and ILM determined in the range $1 < t \le 10^6$ by the first algorithm, and at the ISm candidates and ILM candidates found in the range $10^6 < t \le 10^{16}$ by the second algorithm.

2. Methods of computation

2.1. **General.** The computations were performed on a PC equipped with a 2400 MHz Intel Pentium 4 processor. For $1 < t \le 10^6$, the values of $\zeta(1+it)$ were computed with Mathematica 5.0 (Wolfram Research, Urbana, IL, USA) using the `Zeta` routine, and the ISm and ILM were verified in Delphi 6.0 (Borland, Scotts Valley, CA, USA) with 20-digit precision using the Euler-Maclaurin formula (see Appendix A). For $10^6 < t \le 10^{16}$, the values of $\zeta(1+it)$ were computed in Delphi 6.0 with 20-digit precision using a formula derived from the approximate functional equation [9]

$$\begin{aligned}\zeta(\sigma+it) &= \sum_{n\le x}\tfrac{1}{n^{\sigma+it}} + \tfrac{2^{\sigma+it-1}\pi^{\sigma+it}}{\Gamma(\sigma+it)\cos(\pi(\sigma+it)/2)}\sum_{n\le|t|/(2\pi x)}\tfrac{1}{n^{1-\sigma-it}}\\ &\quad +O(x^{-\sigma})+O(|t|^{\frac{1}{2}-\sigma}y^{\sigma-1}) \quad \text{for } 0\le\sigma\le 1.\end{aligned}$$

Upon setting $\sigma = 1$, $t > 0$, and $x = \sqrt{\frac{t}{2\pi}}$, this gives

$$\zeta(1+it) = \sum_{n\le x}\tfrac{1}{n^{1+it}} + \tfrac{\pi(2\pi)^{it}}{\Gamma(1+it)\cos(\pi(1+it)/2)}\sum_{n\le x} n^{it} + O(t^{-\frac{1}{2}}). \tag{3}$$

Now

$$\cos\tfrac{\pi(1+it)}{2} = -i\sinh\tfrac{\pi t}{2} = -i\tfrac{e^{\pi t/2}-e^{-\pi t/2}}{2}$$

and

$$\begin{aligned}\Gamma(1+it) &= it\Gamma(it) = it\sqrt{2\pi}e^{-it}(it)^{it-1/2}P_S(it)\\ &= \sqrt{2\pi it}(\tfrac{t}{e})^{it}e^{-\pi t/2}P_S(it),\end{aligned}$$

where $P_S(z) = 1 + \frac{1}{12}z^{-1} + \frac{1}{288}z^{-2} - \frac{139}{51840}z^{-3} + ...$ is the Stirling series for the gamma function. Consequently

$$\Gamma(1+it)\cos\tfrac{\pi(1+it)}{2} = \tfrac{\sqrt{\pi t}}{1+i}(\tfrac{t}{e})^{it}(1-e^{-\pi t})(1-\tfrac{i}{12}t^{-1}-\tfrac{1}{188}t^{-2}+...)$$

so that (3) can be rewritten as

$$\zeta(1+it) = \sum_{n\le x} \tfrac{1}{n^{1+it}} + \tfrac{(1+i)(2e)^{it}}{(1-e^{-\pi t})(1-it^{-1}/12-t^{-2}/188+\ldots)} (\tfrac{\pi}{t})^{\frac{1}{2}+it} \sum_{n\le x} n^{it} + O(t^{-\frac{1}{2}}).$$

This formula, with the Stirling series truncated after the term $-t^2/188$ and the error term $O(t^{-1/2})$ omitted, was used for the evaluation of $\zeta(1+it)$ in the range $10^6 < t \le 10^{16}$. Where a higher accuracy was required, e.g. for precise determination of ISm and ILM, the error term $O(t^{-1/2})$ was written in its explicit form as a series (see Theorem 4.16 in [10]), and a suitable number of its terms were evaluated. All the located ISm and ILM were verified in Mathematica 5.0 using the `Zeta` routine.

2.2. **Determination of ISm and ILM in the range $1 \le t \le 10^6$.** The determination of ISm and ILM in the range $1 \le t \le 10^6$ was based on the inequality

$$\left|\tfrac{d}{dt}\left|\zeta(1+it)\right|\right| < \tfrac{1}{2}\log^2 t + 3\log t + \tfrac{27}{8} \quad \text{for } t \ge 1,$$

the proof of which is given in Appendix A. This inequality allows us to apply the maximum-slope principle in verifying whether a certain minimum (resp. maximum) of $|\zeta(1+it)|$ is the smallest (resp. largest) one in a given range $1 \le u_{\min} \le t \le u_{\max}$. Namely, denoting $D(t) := \frac{1}{2}\log^2 t + 3\log t + \frac{27}{8}$, it follows that if $|\zeta(1+it_k)| > a$ for some $a > 0$ and some $t_k > 1$, then also $|\zeta(1+it)| > a$ for $\max(1, t_{k+1}) < t \le t_k$, with $t_{k+1} = t_k - \frac{|\zeta(1+it_k)|-a}{D(t_k)}$. Thus, to show that $|\zeta(1+it)| > a$ for $1 \le u_{\min} \le t \le u_{\max}$, we first evaluate $|\zeta(1+it)|$ at $t_1 = u_{\max}$, then at $t_2 = t_1 - \frac{|\zeta(1+it_1)|-a}{D(t_1)}$, $t_3 = t_2 - \frac{|\zeta(1+it_2)|-a}{D(t_2)}$, ..., until covering the t-range down to $t = u_{\min}$. With obvious modifications, the same approach applies in showing that $|\zeta(1+it)| < b$ for some $b > 0$ and some t-range.

To obtain the list of ISm in the range $1 \le t \le 10^6$, a list of small minima in this t-range was compiled by computing $|\zeta(1+it)|$ at t-values in increments of 0.1 and then determining the local minimum of $|\zeta(1+it)|$ in the neighborhood of each sampled value smaller than 0.7. The ISm were then selected from the computed minima, and the maximum-slope principle described above was used to determine the t-range in which there is no smaller value of $|\zeta(1+it)|$ than the ISm under consideration. The results of this selection are presented in Table 1 of Appendix B.

An analogous approach was used to obtain the list of ILM in the range $1 \le t \le 10^6$: the local maximum of $|\zeta(1+it)|$ was determined in the neighborhood of each sampled value larger than 0.7, the ILM were selected, and the maximum-slope principle was used to determine the t-range in which there is no larger value of $|\zeta(1+it)|$ than the ILM under consideration. The results of this selection are presented in Table 2 of Appendix B.

2.3. **Search for ISm and ILM in the range $10^6 < t \le 10^{16}$.** The approach described in the preceding subsection took roughly a month to cover the range $1 \le t \le 10^6$, which was too slow to allow for an extension to substantially larger t. A more efficient approach was developed based on a consideration of the Euler product

$$\zeta(s) = \prod_p (1-p^{-s})^{-1} \quad \text{for } \Re(s) \ge 1,\ s \ne 1,$$

where p runs through all the primes. Setting $s = 1 + it$ and taking absolute values, we get

$$|\zeta(1+it)| = \prod_p |1 - p^{-1-it}|^{-1} = \prod_p \frac{p}{\sqrt{1-2p\cos(t\log p)+p^2}}.$$

Each factor of this product is a periodic function: the factor corresponding to the prime p has periodic maxima at $t = \frac{2k\pi}{\log p}$ and periodic minima at $t = \frac{(2k-1)\pi}{\log p}$, with the ratio between the maximal and the minimal value equal to $\frac{p+1}{p-1}$. Thus small values of $|\zeta(1+it)|$ may be expected at those t for which $\frac{t\log 2}{2\pi}$, $\frac{t\log 3}{2\pi}$, $\frac{t\log 5}{2\pi}$, ... have a fractional part close to $\frac{1}{2}$, so that the first factors in the above product are close to their minima. Similarly, large values of $|\zeta(1+it)|$ may be expected at those t for which $\frac{t\log 2}{2\pi}$, $\frac{t\log 3}{2\pi}$, $\frac{t\log 5}{2\pi}$, ... are close to an integer, so that the first factors in the above product are close to their maxima. The ISm were thus searched for in the vicinity of the points $t_k^* := \frac{(2k-1)\pi}{\log 2}$, and an additional selection was made based on the absolute deviation of $\frac{\log 3}{2\pi}t_k^* - \frac{1}{2}$, $\frac{\log 5}{2\pi}t_k^* - \frac{1}{2}$, $\frac{\log 7}{2\pi}t_k^* - \frac{1}{2}$, ..., from the nearest integer. Similarly, the ILM were searched for in the vicinity of the points $t_k := \frac{2k\pi}{\log 2}$, and an additional selection was made based on the absolute deviation of $\frac{\log 3}{2\pi}t_k$, $\frac{\log 5}{2\pi}t_k$, $\frac{\log 7}{2\pi}t_k$, ..., from the nearest integer.[1] For the range $10^n < t \le 10^{n+1}$, the selection criteria were set so that none of the ISm and ILM found in the range $10^{n-3} < t \le 10^n$ would have been missed using the same criteria. The primes up to $p = 13$ were used for selection in the range $10^6 < t \le 10^9$, the primes up to $p = 23$ in the range $10^9 < t \le 10^{12}$, and the primes up to $p = 37$ in the range $10^{12} < t \le 10^{16}$. As t increased, the selection criteria for the ILM candidates (the maximum allowed absolute deviations of $\frac{\log p}{2\pi}t_k$ from an integer) set according to these rules were becoming more stringent rather rapidly, which resulted in a significant speed increase of the search algorithm. With the selection criteria for the ISm candidates (the maximum allowed absolute deviations of $\frac{\log p}{2\pi}t_k^* - \frac{1}{2}$ from an integer) this was much less pronounced, so that a larger number of candidates had to be verified, resulting in a slower search algorithm. Due to this, above $t = 10^9$ the searches for the ILM and ISm candidates were run separately. The speed difference between the two algorithms was so substantial that in approximately fourteen months of computation, the search for the ILM candidates already reached $t = 10^{16}$ while the search for the ISm candidates had only covered the range up to $t = 10^{12}$. The data given in Appendix B thus contain the ILM candidates found up to 10^{16}, and the ISm candidates found up to 10^{12}.

3. Results and discussion

Figure 2 shows the estimates of $\beta(1)$ at the ILM in the range $1 \le t \le 10^6$, ILM candidates found in the range $10^6 < t \le 10^{16}$, ISm in the range $1 \le t \le 10^6$, and ISm candidates found in the range $10^6 < t \le 10^{12}$, with solid circles corresponding to the ILM and ILM candidates, and hollow ones to the ISm and ISm candidates. As these data reveal, the behavior of these estimates of $\beta(1)$ remains similar throughout the investigated t-range: as t increases, the estimates at the ILM and ILM candidates tend to increase slowly, gradually reaching values somewhat above 0.64, while the

[1] Although based on a different formula for $\zeta(s)$ (as the Euler product diverges for $\Re(s) < 1$) this same approach can also be used efficiently in locating large values of $|\zeta(\frac{1}{2}+it)|$; see [11].

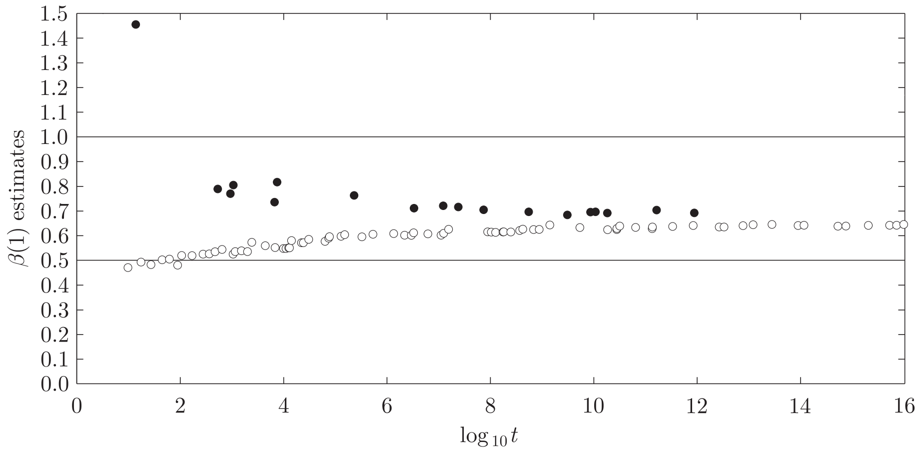

FIGURE 2

estimates at the ISm and ISm candidates tend to decrease slowly, gradually reaching values somewhat below 0.70.

It seems reasonable to assume that asymptotically, the two sets of estimates of $\beta(1)$ should both converge to the actual value of $\beta(1)$. To date, the only rigorous insight into this asymptotic behavior is provided by the recent work of Granville and Soundararajan published in an arXiv preprint in 2005 [8]. This work represents a crucial improvement with respect to Levinson's inequalities in (1) in the sense that one of the theorems characterizes, to some extent, the asymptotic behavior of the estimates of $\beta(1)$ at the largest $|\zeta(1+it)|$, and one of the conjectures – if true – would also do this for the estimates of $\beta(1)$ at the smallest $|\zeta(1+it)|$. In the subsequent paragraphs the numerical results obtained in this paper will be analyzed and discussed together with the implications of these theorems and conjectures.

Theorem 2 in [8] asserts that there exist arbitrarily large t such that

$$|\zeta(1+it)| \geq e^{\gamma}(\log\log t + \log\log\log t - \log\log\log\log t + O(1)),$$

which can be rewritten as

$$\frac{|\zeta(1+it)|}{2e^{\gamma}\log\log t} \geq \frac{1}{2} + \frac{\log\log\log t - \log\log\log\log t + O(1)}{2\log\log t}. \tag{4}$$

Thus if $\beta(1)$ is actually the smallest possible, i.e. $\beta(1) = \frac{1}{2}$, then the estimates of $\beta(1)$ at the ILM should in general approach this value from above. This would imply that the observed behavior of the estimates of $\beta(1)$ at the ILM changes after a large-scale increase that seems to persist at least up to $t = 10^{16}$, and the estimates of $\beta(1)$ at the ILM start to decrease again. This scenario may not be too unlikely provided that (4) is very sharp, since for $t = 10^{16}$ its right-hand side, with the omission of the term $O(1)$, equals 0.643..., while the estimate of $\beta(1)$ at the largest ILM found in this study at $t = 9.46... \times 10^{15}$ (see Table 3 in Appendix B) equals 0.644....

In [8] it is conjectured that (4) can be improved further to

$$\frac{|\zeta(1+it)|}{2e^{\gamma}\log\log t} \geq \frac{1}{2} + \frac{\log\log\log t - C + o(1)}{2\log\log t} \tag{5}$$

with $C = 0.0893....$ This finds slightly less support in the numerical data presented in this paper, since for $t = 10^{16}$ the right-hand side of (5), with the omission of the term $o(1)$, equals 0.665....

The above considerations only apply if $\beta(1) = \frac{1}{2}$. If this is not the case, it is possible that the observed large-scale behaviors of the two sets of estimates of $\beta(1)$ continue also asymptotically, so that the actual value of $\beta(1)$ is somewhere between 0.64 and 0.70, and perhaps close to $\frac{2}{3}$. Moreover, although it seems less likely, it is impossible to refute the possibility that $\beta(1)$ is actually above 0.70 (or even above 1, provided that the Riemann hypothesis is false), which would imply that after a large-scale decrease, the estimates of $\beta(1)$ at the ISm must start to increase again.

The insight provided by numerical data would improve significantly if theoretical developments would lead to an inequality characterizing – in analogy to (4) and with similar sharpness – the small values of $|\zeta(1+it)|$. Currently the strongest result of this kind is implied by Theorem 1 in [8], from which it follows that there exist arbitrarily large values of t such that

$$|\zeta(1+it)| \le \frac{\zeta(2)}{e^{\gamma}(\log\log t - O(1))},$$

which can be rewritten as

$$\frac{\zeta(2)}{2|\zeta(1+it)|e^{\gamma}\log\log t} \ge \frac{1}{2} - \frac{O(1)}{\log\log t}. \tag{6}$$

The estimates of $\beta(1)$ at the ISm and the ISm candidates found in the range up to $t = 10^{12}$ are all larger than $\frac{1}{2}$, and considerably so, which suggests that there is some space for improvement of this inequality. In [8] it is conjectured that (6) can be strengthened further to

$$\frac{\zeta(2)}{2|\zeta(1+it)|e^{\gamma}\log\log t} \ge \frac{1}{2} + \frac{\log\log\log t - C + o(1)}{2\log\log t}, \tag{7}$$

with the same value of C as in (5). This would imply that if $\beta(1) = \frac{1}{2}$, then also the estimates of $\beta(1)$ at the ISm should in general approach this value from above. Still, while the right-hand sides of (5) and (7) are formulated identically, the numerical data presented in this paper show that at least for small t, the estimates of $\beta(1)$ at the smallest $|\zeta(1+it)|$ behave rather differently from those at the largest $|\zeta(1+it)|$. This does not necessarily suggest that either (5) or (7) is false asymptotically, but if they are both true, then it appears that the explicit functional forms of the terms $o(1)$ in these two inequalities could differ quite considerably.

Appendix A. To show that

$$\left|\frac{d}{dt}|\zeta(1+it)|\right| < \frac{1}{2}\log^2 t + 3\log t + \frac{27}{8} \quad \text{for } t \ge 1$$

we proceed from the Euler-Maclaurin summation formula (see (3.5.3) in [10])

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + s\int_N^{\infty} \frac{1/2 - \{x\}}{x^{s+1}}dx + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} \quad \text{for } \Re(s) > 0,\ s \ne 1.$$

Differentiation yields

$$\frac{d}{ds}\zeta(s) = -\sum_{n=2}^{N} \frac{\log n}{n^s} + \int_N^{\infty} \frac{1/2 - \{x\}}{x^{s+1}}(1 - s\log x)dx - \frac{N^{1-s}\log N}{s-1} - \frac{N^{1-s}}{(s-1)^2} + \frac{N^{-s}\log N}{2}$$

and by taking absolute values, we obtain

$$\begin{aligned}
&\left|\tfrac{d}{ds}\zeta(s)\right| \\
&\le \sum_{n=2}^{N} \tfrac{\log n}{n^\sigma} + \int_N^\infty \tfrac{|1/2-\{x\}|}{x^{\sigma+1}} |1-s\log x|\,dx + \tfrac{N^{1-\sigma}\log N}{\sqrt{(\sigma-1)^2+t^2}} + \tfrac{N^{1-\sigma}}{(\sigma-1)^2+t^2} + \tfrac{N^{-\sigma}\log N}{2} \\
&< \sum_{n=2}^{N} \tfrac{\log n}{n^\sigma} + \tfrac{1}{2}\int_N^\infty \tfrac{1+\sqrt{\sigma^2+t^2}\log x}{x^{\sigma+1}}dx + \tfrac{N^{1-\sigma}\log N}{\sqrt{(\sigma-1)^2+t^2}} + \tfrac{N^{1-\sigma}}{(\sigma-1)^2+t^2} + \tfrac{N^{-\sigma}\log N}{2} \\
&< \sum_{n=2}^{N} \tfrac{\log n}{n^\sigma} + \tfrac{N^{-\sigma}}{2\sigma} + \tfrac{(\sigma+t)N^{-\sigma}(\sigma\log N+1)}{2\sigma^2} + \tfrac{N^{1-\sigma}\log N}{\sqrt{(\sigma-1)^2+t^2}} + \tfrac{N^{1-\sigma}}{(\sigma-1)^2+t^2} + \tfrac{N^{-\sigma}\log N}{2}.
\end{aligned}$$

Taking $\sigma = 1$, $t \ge 1$, and $N = [t]$, we get

$$\begin{aligned}
\left|\tfrac{d}{dt}\zeta(1+it)\right| &< \sum_{n=2}^{[t]} \tfrac{\log n}{n} + \tfrac{1}{2[t]} + \tfrac{(1+t)(\log t+1)}{2[t]} + \tfrac{\log t}{[t]} + \tfrac{1}{t^2} + \tfrac{\log t}{2[t]} \\
&< \int_1^t \tfrac{\log x}{x}dx + \max_{1\le x\le t} \tfrac{\log x}{x} + \tfrac{(4+t)\log t}{2[t]} + \tfrac{2+t}{2[t]} + \tfrac{1}{t^2} \\
&< \tfrac{1}{2}\log^2 t + \tfrac{1}{e} + 3\log t + 2 + 1 \\
&< \tfrac{1}{2}\log^2 t + 3\log t + \tfrac{27}{8}.
\end{aligned}$$

On the half-line $t \ge 1$, the function $\zeta(1+it)$ is analytic and has no zeros, so that $|\zeta(1+it)|$ is differentiable. Therefore

$$\begin{aligned}
\left|\tfrac{d}{dt}\,|\zeta(1+it)|\right| &= \left|\lim_{h\to 0} \tfrac{|\zeta(1+i(t+h))|-|\zeta(1+it)|}{h}\right| \\
&\le \left|\lim_{h\to 0} \tfrac{\zeta(1+i(t+h))-\zeta(1+it)}{h}\right| = \left|\tfrac{d}{dt}\zeta(1+it)\right|,
\end{aligned}$$

and the inequality follows.

Appendix B.

TABLE 1. The ISm in the range $1 \le t \le 10^6$

T	$\lvert\zeta(1+iT)\rvert$	subrange with no smaller $\lvert\zeta(1+it)\rvert$
2.922710	0.637686	$t < 13.194364$
14.118038	0.326035	$t < 540.337342$
540.449180	0.318453	$t < 946.836877$
946.930508	0.311961	$t < 1083.080867$
1083.242770	0.295304	$t < 6740.196042$
6740.297163	0.288634	$t < 7563.356634$
7563.561158	0.258500	$t < 231498.768858$
231498.924322	0.240996	$t \le 10^6$

TABLE 2. The ILM in the range $1 \le t \le 10^6$

T	$\lvert\zeta(1+iT)\rvert$	subrange with no larger $\lvert\zeta(1+it)\rvert$
9.986678	1.394626	$t < 16.226758$
17.838107	1.854791	$t < 26.956117$
27.716063	2.060971	$t < 44.874060$
45.625818	2.394259	$t < 62.612196$
63.062221	2.553839	$t < 90.588000$
90.728555	2.574261	$t < 108.479241$
108.976510	2.858658	$t < 171.435901$
171.767363	3.023872	$t < 280.477613$
280.806738	3.231898	$t < 371.335278$
371.545472	3.329659	$t < 480.169690$
480.402875	3.463988	$t < 651.983673$
652.218097	3.616653	$t < 1069.321682$
1069.377640	3.626947	$t < 1178.282699$
1178.452319	3.724869	$t < 1549.859674$
1550.022297	3.822850	$t < 2030.404245$
2030.510518	3.868193	$t < 2447.359710$
2447.632331	4.188011	$t < 4477.992038$
4478.089070	4.235902	$t < 6925.543599$
6925.631647	4.279233	$t < 10025.497032$
10025.587061	4.327658	$t < 11204.128809$
11204.193388	4.353134	$t < 12645.042976$
12645.129488	4.400517	$t < 13125.434639$
13125.468191	4.407712	$t < 14303.781996$
14303.977899	4.662864	$t < 22299.036031$
22299.089892	4.683982	$t < 24329.581964$
24329.631240	4.701992	$t < 30774.812217$
30774.955309	4.861092	$t < 63751.784306$
63751.873448	4.930674	$t < 74955.902446$
74956.029228	5.077058	$t < 77403.634482$
77403.711489	5.132144	$t < 130060.456341$
130060.560181	5.241710	$t < 152359.658483$
152359.749856	5.328706	$t < 328768.163674$
328768.233490	5.384846	$t < 534573.561546$
534573.681207	5.560790	$t \le 10^6$

Table 3. The ISm candidates (left) and ILM candidates (right) found in the range $10^6 < t \leq 10^{16}$

t	$\|\zeta(1+it)\|$
3316785.109381	0.239941
12140638.479404	0.229565
23845694.162806	0.227883
73440455.332597	0.226505
544499269.334798	0.221138
3025392287.432489	0.219226
8499373753.445245	0.212313
10598231107.361198	0.211401
17949023567.845834	0.211390
160088356577.999627	0.202176
855204807584.978724	0.201544

t	$\|\zeta(1+it)\|$
1345367.796178	5.731671
2186410.518605	5.739152
2939652.697912	5.780885
3268420.884302	5.891247
6155416.652199	5.940139
11026769.628406	5.968216
12372137.481160	6.057685
15457423.710003	6.249120
87568424.958195	6.363699
102805259.027870	6.370154
124570459.065104	6.383583
173723252.245254	6.439189
178900422.227160	6.449103
244946055.640466	6.479234
363991205.177417	6.587479
418878041.151948	6.659547
673297382.180989	6.696291
868556070.988342	6.726610
1387123309.982725	6.974882
5272517912.834557	6.999372
18168214001.678085	7.020629
27279224693.800531	7.066738
27331684151.584170	7.118618
31051083602.362376	7.236757
62792807608.659331	7.244279
131443859639.680785	7.249398
133159989048.419665	7.329600
326473979757.409463	7.431172
812980259631.158140	7.555727
2589877332690.837705	7.574375
3210707929490.460259	7.592861
7466630414566.965400	7.723197
11689156552576.453957	7.805226
27070203555136.549316	7.878501
86087778561794.083555	7.914445
111859624805509.264416	7.946646
506044483208373.891103	8.000510
719342003522637.683814	8.030843
1954451350021854.506285	8.128581
5028281246314787.290881	8.194452
6869284037933465.204919	8.219602
9460455379268814.248721	8.279901

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